CRITICISMS.

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"THE tensor of a quaternion vector ρ, denoted by $T\rho$, is the length of the vector expressed in terms of the unit of length and without regard to direction, and hence is an abstract number, signless as those used by a child in counting, but called positive in contradistinction to the negatives of algebra. Multiplied into a vector of unit length it gives to it the length of ρ, and when the unit vector has the direction of ρ, or is $U\rho$ the result is ρ itself, viz., we have $T\rho\, U\rho = \rho$; and this operation performed upon the unit vector $U\rho$ to produce ρ may be conceived of as an act of *tension, extension* or *stretching*. But when ρ itself is operated upon by a scalar b to produce the *new* vector $b\rho$, or by the scalar c and the inversor $(-)$ to produce the *new* vector $-c\rho$, the first by a stretching of ρ, the second by a stretching and an inversion, we must not suppose that in the one case $T\rho = b$, in the other $T\rho = c$, — while to write $T\rho = -c$ is to *include the very thing* which the characteristic T was devised to *ex*clude, and to which that other symbol, U, has been appropriated, viz., the *versor* or *directional* element $(-)$. b and c are factors of the tensors of the vectors *by them produced*, viz., $T(b\rho) = bT\rho$, $T(-c\rho) = T(c\rho) = cT\rho$. In fine, the tensor which produces ρ from a unit vector is the tensor of ρ, $T\rho$, in the technical sense in which Hamilton employs the term Tensor and the symbol T; but the tensor which produces from ρ some *other* vector is *not* the tensor of ρ. In the one case ρ is the *result* of the stretching or tension, in the other the *subject*, and between these two cases it is necessary and easy to distinguish."

The above was sent to Prof. Wood immediately upon the publication of §9, p. 36 of the present volume of the ANALYST, and I insert it here because I deem the correction on p. 127 neither sufficiently general nor adequately emphatic.

At p. 69, line 5 we read: "the letter U may also be used to denote a unit vector; thus $U\beta$ is a unit vector parallel to vector β. As a versor it implies that β has been turned from some arbitrary direction into the given one. At p. 128 the latter sentence is expunged, because "it conflicts with the definition given by Hamilton, which is '$U\beta$ is *the versor of a right quotient*.'" Now, although there is nothing in Hamilton to *justify* the expunged statement, there is, on the other hand, nothing there with which it *conflicts*. The additional meaning thereby assigned to the symbol U is merely *worthless*. Hamilton calls a unit vector a *versor* from *analogy* and for *convenience*, since

we have $Uq = U\frac{\beta}{\alpha} = \frac{U\beta}{U\alpha}$; but he afterward shows that a vector, say β, may be usefully equated to the *right quotient* of which it is the *index*, and $U\beta$ to the corresponding *right radial;* furthermore, that every unit vector may be regarded as the axis of a *quadrantal rotation*, and that the vector may be considered as itself a *quaternion*, of which the tensor of the vector is the tensor, the *unit vector* $U\beta$, say, its *versor*. Hence U does not imply anything whatever, but plainly *expresses*, *not* that "β has been turned from some arbitrary direction into the given one", but that β itself has, as a quaternion, power to turn *other* vectors from *their* directions, this *turning* power being resident in $U\beta$. It will be seen that the distinction is that between the *actor* and *the thing acted upon.*

Turning now to Prof. Wood's treatment of the important symbols i, j, k, we find on p. 67 the equations $k = \frac{j}{i}$ (3), $k = \frac{-i}{j}$ (4), $-k = \frac{i}{j}$ (5), and thereafter is the remark: "A comparison of equations (3) and (5) shows that a reciprocal of the fraction *changes the sign* of the vector axis, instead of producing its reciprocal." Of course we have $\frac{i}{j} = -k = \frac{1}{k}$, which *is* the reciprocal of k, the "axis" of $j \div i$, and Prof. W. attends to the matter at top of p. 128; but he there restricts the necessity for the emendation to the case "When the direction of rotation of the fraction and its reciprocal are in opposite senses." It is matter of easy and direct perception, as well as of early definition, that the direction of rotation of a fraction is *always* opposite to that of its reciprocal. Again, on p. 68, the equation $kk = -1$ is obtained from (3) and (4) by the transformation

$$kk = \frac{j}{i} \cdot \frac{-i}{j} = \frac{-i}{i} = -1.$$

Now by definition

$$\frac{\beta}{\alpha} \cdot \alpha [= \beta\alpha^{-1}\alpha] = \beta, \frac{\gamma}{\beta} \cdot \frac{\beta}{\alpha} [= \gamma\beta^{-1}\beta\alpha^{-1} = \gamma\alpha^{-1}] = \frac{\gamma}{\alpha},$$

but *not* $\alpha . \frac{\beta}{\alpha} [= \alpha\beta\alpha^{-1}] = \beta$, *nor* $\frac{\beta}{\alpha} \cdot \frac{\gamma}{\beta} [= \beta\alpha^{-1}\gamma\beta^{-1}] = \frac{\gamma}{\alpha}$; and hence we should write $\frac{j}{i} \cdot \frac{-i}{j} [= -ji^{-1}ij^{-1} = -jj^{-1}] = -\frac{j}{j} = -1$, but *never* $\frac{j}{i} \cdot \frac{-i}{j} = \frac{-i}{i}$. The j's in Prof. W.'s transformation are *non-adjacent*—to combine them is to do away with the non-commutative principle of quaternion (or vector) multiplication, and this, we may say, is to do away with quaternions themselves. Had Prof. W. desired to pass through the form $\frac{-i}{i}$ he should have written $kk = \frac{-i}{j} \cdot \frac{j}{i} [= -ij^{-1}ji^{-1} = -ii^{-1}] = \frac{-i}{i}$

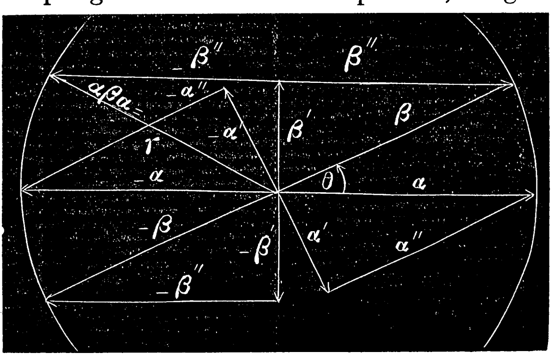
$= -1$, which is permissible. Owing to the *scalar* character of the product, which permits of cyclic permutation under the characteristic S, or more shortly, to the *complanarity* of the factor quaternions, the *result* arrived at is *correct;* but the *process* is not legitimate and is especially unjustifiable in a paper avowedly expository of the principles of this calculus.

On p. 123, Prof. W. says, "In division, the versor operating on the divisor line is conceived to turn it, positively, about the axis of the versor $(+i)$ through an angle equal to that of the versor (θ) to coincide in direction with the dividend line. In multiplication the versor operating on the multiplier line [as α, eq. (27)] is conceived to turn it in a positive direction about the axis of the versor $(+i)$ through an angle equal to the *supplement* of the angle of the versor $[\pi - (\pi - \theta) = \theta]$, making it coincide in direction with the multiplicand line (as β)." Prof. W. cites p. 85 of the Lectures for the above, but I can assure the readers of the ANALYST that Hamilton does not deserve the credit of it. In the "division"

$$\frac{\beta}{\alpha} = \cos\theta + i\sin\theta,$$

the versor $\cos\theta + i\sin\theta$, so far from "operating upon the divisor line" (α) in reality *does not operate upon any line whatever, does not act at all,* but simply *constitutes the conception* $\beta \div \alpha$. So soon as the equivalency of the vector and the right part of the quaternion is established, we may change the "quotient" $\frac{\beta}{\alpha}$ into the "product" $\beta . \frac{1}{\alpha} = -\beta\alpha$, and *then* we have a *bonafide* "operation", viz., β and the inversor $(-)$ operate on α; and for all such products of complanar unit vectors a multitude of simple constructions may be given.

As a specimen, let the figure be a diametral section of the unit sphere by the plane of the paper. Adopting left hand rotation as positive, imagine the unit vector i drawn perpendicular to this section and in front. Resolve α into α' and α'', the former perpendicular, the latter parallel to β, &c.



Then we have by inspection,

$$\frac{\beta}{\alpha} = -(\beta\alpha) = -(\beta' + \beta'')\alpha$$

$$= -(\beta'\alpha) - (\beta''\alpha) = -[(-i)\sin\theta] + (\alpha^2\cos\theta) = i\sin\theta + \cos\theta,$$

$$\frac{\beta}{\alpha} = -(\beta\alpha) = -\beta(\alpha'+\alpha'') = -(\beta\alpha')-(\beta\alpha'') = -(-i)\sin\theta$$
$$-(\beta^2\cos\theta) = i\sin\theta + \cos\theta,$$

$$\frac{\beta}{\alpha} = (-\beta)\alpha = (-\beta)(\alpha'+\alpha'') = (-\beta)\alpha'+(-\beta)\alpha'' = i\sin\theta - \beta^2\cos\theta$$
$$= i\sin\theta + \cos\theta,$$

&c., &c., &c.,

$$\alpha\beta = (\alpha'+\alpha'')\beta = \alpha'\beta+\alpha''\beta = i\sin\theta + \beta^2\cos\theta = i\sin\theta - \cos\theta,$$

$$\alpha\beta = \alpha(\beta'+\beta'') = \alpha\beta'+\alpha\beta'' = i\sin\theta + \alpha^2\cos\theta = i\sin\theta - \cos\theta.$$

$$\frac{\beta}{\alpha}.\alpha = (i\sin\theta + \cos\theta)\alpha = i\,\alpha\sin\theta + \alpha\cos\theta = \beta' + \beta'' = \beta.$$

$$\alpha\beta.\alpha = (i\sin\theta - \cos\theta)\alpha = i\,\alpha\sin\theta + (-\alpha)\cos\theta = \beta' + (-\beta'') = \gamma,$$

and so on interminably.

It is easily seen that the product of an *even* number of complanar vectors is a *quaternion* in the same plane, that of an *odd* number, a *vector* complanar with the factors, and that for every such *odd* product we have $S\alpha\beta\gamma \ldots . \omega = 0$, special cases of which are the constantly recurring formulæ $S\alpha = 0$, $S\alpha\beta\gamma = 0$. But these matters must be left to expositors of the method.

To recapitulate results: It is seen that in the *division* $\beta \div \alpha$ the versor $\cos\theta + i\sin\theta$ does not operate at all; in the equivalent *multiplication* $\beta.\frac{1}{\alpha}$, β and the inversor (—) operate upon α to produce the versor $\cos\theta + i\sin\theta$. Let this be compared with the first part of Prof. W's statement.

In the multiplication $\alpha\beta$, the versor $-\cos\theta + i\sin\theta$, so far from operating upon the multiplier line α, is the *result* of α operating upon β —the multiplier line upon the multiplicand line, as it should be.

It is possible that Prof. W. had in mind that

$$\frac{\beta}{\alpha}.\alpha = (\cos\theta + i\sin\theta)\alpha = \beta,$$

which is matter of definition; but this is quite a different affair from the division $\beta \div \alpha$; and, on the other hand, we do *not* have

$$\alpha\beta.\alpha = (-\cos\theta + i\sin\theta)\alpha = \beta.$$

In fact this versor $-\cos\theta + i\sin\theta$, when it comes to operate upon α (as a *multiplicand*, of course) turns the latter, as shown by the last of the above equations, through an angle $\pi - \theta$ into coincidence with γ, what Hamilton would call the *reflection* of β with respect to the line β'.